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Rapport de stage

Catégories $*$-autonomes sans unité

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1 Introduction

Le contexte général

La théorie de la démonstration est une branche de la logique mathématique fondée par David Hilbert au début du XX^{e} siècle dans le but de résoudre le problème de la cohérence des mathématiques en développant une théorie permettant de comparer les preuves. La découverte de la correspondance de Curry-Howard dans les années 1960 a établi un lien nouveau entre logique et informatique : les preuves deviennent des programmes dont le type est la proposition à démontrer. Dès lors, la théorie de la démonstration s'est développée en étroit rapport avec l'informatique théorique, notamment avec le lambda-calcul et, à partir de 1987, avec la logique linéaire (voir [Gir87]). De plus, la logique linéaire et plus particulièrement son fragment multiplicatif (MLL) et son fragment multiplicatif sans unité (MLL^-), sont dotés d'une théorie d'identification des preuves particulièrement puissante que sont les réseaux de preuve.

Indépendamment, en 1969 Lambek propose une identification des preuves basée sur des diagrammes commutatifs dans des catégories qui sont vues comme des systèmes déductifs : deux preuves sont les mêmes si elles correspondent au même morphisme dans la catégorie. Il applique ceci à la logique intuitionniste et aux catégories cartésiennes closes (voir [Lam68] et [Lam69]). Plus tard, Lafont dans [Laf88] obtient une correspondance entre la logique linéaire et les catégories $*$-autonomes (introduites dans [Bar79]). Mieux encore, pour MLL, la catégorie des séquents à deux formules et des réseaux de preuve sans coupures, est la catégorie $*$-autonome libre générée par l'ensemble des formules atomiques (voir [LS06]). Ces résultats s'avèrent très utiles ; ils fournissent un nouveau cadre de travail pour la logique, un cadre purement catégorique qui permet d'obtenir de nouveaux critères d'identification des preuves.

Le problème étudié

Pour MLL^- un résultat similaire semble être presque vrai (voir [Blu93]), mais la correspondance entre catégories $*$-autonomes et réseaux de preuve de MLL^- est imparfaite car, si les unités I et $\perp$ jouent un rôle fondamental dans la définition d'une catégorie $*$-autonome, les réseaux de preuve de MLL^- sont eux, par définition, dépourvus d'unités. Plus précisément, si dans le cas de MLL une preuve de $\vdash A$, i.e. d'une formule A à partir de l'unité I, correspond dans la catégorie $*$-autonome libre à un morphisme allant de l'unité vers A, dans le cas de MLL^- on doit trouver un morphisme de la catégorie $*$-autonome sans unité libre qui corresponde à une preuve de $\vdash A$, i.e. de A à partir de rien.

Il paraît naturel de chercher à établir une notion de catégorie $*$-autonome sans unités de sorte que les réseaux de preuve (sans coupure) de MLL^- forment une telle catégorie libre. Ceci nous permettrait, par exemple, de traduire des preuves de MLL^- directement dans la catégorie $*$-autonome libre. Deux preuves de MLL^- ainsi traduites correspondraient au même morphisme dans la catégorie si et seulement si elles donnent lieu, après normalisation, au même réseau de preuve. Ainsi, l'on pourrait travailler de façon équivalente au niveau de la catégorie $*$-autonome libre, tout comme au niveau du système formel de la logique.

Ce n'est que très récemment que des tentatives pour définir les catégories $*$-autonomes sans unité ont été faites, notamment dans [LS05], [HHS05] et [DP07]. Aucune de ces propositions ne semble avoir été menée à bout.

Lors de ce stage nous nous sommes penchés sur la définition de catégorie $*$-autonome sans unité suggérée par F. Lamarche et L. Straßburger dans [LS05] (p. 2 à 4). Notre travail a consisté dans un premier temps à comprendre et à formaliser cette définition, puis à l'étudier et à l'étendre dans le but précis que les réseaux de preuve de MLL^- forment une telle catégorie libre.

La contribution proposée

Point de vue catégorique

Nous partons d'une catégorie $\mathscr{C}$ munie d'un bifoncteur produit tensoriel associatif, symétrique et clos. L'idée de la définition de catégorie $*$-autonome libre sans unités repose sur le lemme de Yoneda. Ce dernier permet de voir les foncteurs allant de $\mathscr{C}$ dans la catégorie Set des ensembles et fonctions, comme des objets «généralisés» de $\mathscr{C}$. Nous appelons ces foncteurs *objets virtuels* de $\mathscr{C}$. Ainsi, une catégorie $*$-autonome sans unité est dépourvue d'unités mais est pourvue d'un certain foncteur de $\mathscr{C}$ dans Set (appelé *unité virtuelle* de $\mathscr{C}$ et noté $\mathbb{I}$). Ce foncteur interagit avec les objets de $\mathscr{C}$ à travers un bifoncteur (appelé *produit tensoriel virtuel* et noté $-\otimes_v -$) et un isomorphisme $\tilde{\lambda}$ d'unité virtuelle à gauche. Ce travail est présenté à la section 3.1.

L'axiomatisation d'une telle catégorie $*$-autonome sans unité, avec unité virtuelle, présente des similitudes avec celle d'une catégorie $*$-autonome. Nous avons montré formellement les liens entre ces deux axiomatisations à la section 3.2. L'existence de ces liens permet notamment d'utiliser des résultats obtenus pour MLL (car les réseaux de MLL forment une catégorie $*$-autonome libre) dans le cas de MLL^-.

Point de vue logique

Les réseaux de preuve de MLL^- forment une telle catégorie $*$-autonome sans unité. L'idée est qu'une preuve de $\vdash A$, i.e. d'une formule A de MLL^- à partir de rien, est un élément de l'ensemble d'arrivée du foncteur unité virtuelle (qui va de $\mathscr{C}$ dans Set). Lorsque l'unité n'est pas en jeu tout se passe comme dans le cas des réseaux de MLL.

Les arguments en faveur de sa validité

Mais, si les réseaux de MLL^- forment une telle catégorie $*$-autonome sans unité, la catégorie de ces réseaux n'est pas la catégorie $*$-autonome libre sans unité. Nous montrons ceci à travers un contre-exemple (voir section 5). Plus précisément, cette définition de catégories $*$-autonomes sans unité induit une définition de foncteur entre de telles catégories qui semble problématique.

Le bilan et les perspectives

Lors de ce stage, nous avons formalisé la définition de catégorie $*$-autonome sans unité présentée dans [LS05] et montré quelques propriétés fondamentales de celle-ci. Cependant, nous avons montré qu'une telle catégorie libre ne correspond pas à MLL^-. Il faut donc se pencher sur le problème que cette définition pose. Plus précisément il faudrait essayer de trouver une nouvelle définition de foncteur $*$-autonome sans unité de sorte qu'il préserve les unités virtuelles.

In this paper we use the following notations :

- Set denotes the category of sets and functions between sets;
- $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$ denotes the category of functors between the categories $\mathscr{C}$ and $\mathscr{D}$ and natural transformations between those functors;
- $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ denotes the set of all morphisms from X to Y in the category $\mathscr{C}$;
- given an object X in a category, 1_X denotes the identity morphism on X;
- $\mathsf{PN}^-(\mathscr{A})$ denotes the category where objects are formulas of MLL^- on the set of propositional variables $\mathscr{A} = \{a, b, c, \ldots\} \cup \mathscr{A}^* = \{a^*, b^*, c^*, \ldots\}$ and a morphism from A to B is a cut-free MLL^- proof-net of the formula $\vdash A \multimap B$;
- $(*)$ denotes the terminal category with one morphism and one object.

2 Categorical preliminaries

2.1 Symmetric monoidal closed categories

Definition 2.1. A *category with symmetric tensor* is a category $\mathscr{C}$ equipped with a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C,$$
$$\sigma_{A,B} : A \otimes B \cong B \otimes A,$$

natural in A, B, C in $\mathscr{C}$, such that :

1. we have $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A \otimes B}$ for all A, B in $\mathscr{C}$;
2. the pentagonal diagram

$$\begin{tikzcd} A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \arrow[d, "\alpha\otimes 1"] \\ A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"] & & (A \otimes (B \otimes C)) \otimes D \end{tikzcd}$$

   commutes for all A, B, C and D in $\mathscr{C}$;
3. the hexagonal diagram

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "1\otimes\sigma"'] & (A \otimes B) \otimes C \arrow[r, "\sigma"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\ A \otimes (C \otimes B) \arrow[r, "\alpha"] & (A \otimes C) \otimes B \arrow[r, "\sigma\otimes 1"] & (C \otimes A) \otimes B \end{tikzcd}$$

   commutes for all A, B and C in $\mathscr{C}$.

Definition 2.2. A category with symmetric tensor $\mathscr{C}$ is said to be *closed* if for each object B of $\mathscr{C}$, the functor $(-) \otimes B$ has a right adjoint.

We denote by $B \multimap (-)$ the right adjoint of the functor $(-) \otimes B$. By adjunction, there is an isomorphism

$$\Phi_{A,B,C} : \mathsf{Hom}_{\mathscr{C}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathscr{C}}(A, B \multimap C) \tag{1}$$

natural in A and C in $\mathscr{C}$. By the theorem of *adjunctions with a parameter* (see [Lan98], p.102), we have that $(-) \multimap (-)$ is a bifunctor from $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ to $\mathscr{C}$ and that the isomorphism (1) is natural in all three variables A, B and C.

Remark 2.3. The bifunctor $(-) \multimap (-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ induces a functor

$$H^{(-)} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathscr{C})$$

that takes $X \in \mathscr{C}$ to $H^X = X \multimap (-)$ in $\mathsf{Fun}(\mathscr{C}, \mathscr{C})$.

Remark 2.4. The isomorphism $\Phi_{A,B,C}$ is natural in all three variables A, B and C means that $\Phi_{A,B,C}$ transforms every commuting diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "g"] & C \arrow[d, "h_C"] \\
A' \otimes B' \arrow[u, "h_A \otimes h_B"] \arrow[r, "f"'] & C'
\end{tikzcd}$$

in a commuting diagram

$$\begin{tikzcd}
A \arrow[r, "\Phi_{A,B,C}(g)"] & B \multimap C \arrow[d, "h_B \multimap h_C"] \\
A' \arrow[u, "h_A"] \arrow[r, "\Phi_{A',B',C'}(f)"'] & B' \multimap C'.
\end{tikzcd}$$

for every morphism $h_A : A' \to A$, $h_B : B' \to B$ and $h_C : C \to C'$. Such diagrams are said to be *adjoint diagrams* of each other.

Remark 2.5. In a closed category with symmetric tensor, for all objects A, B, C, one can consider:

- the *evaluation morphisms*

  $$\mathsf{ev}_{A,B} = \Phi^{-1}_{A \multimap B, A, B}(1_{A \multimap B}) : (A \multimap B) \otimes A \to B$$

  corresponding by adjunction with the identity on $A \multimap B$;

- the *composition morphisms* $\mathsf{c}_{A,B,C} : (A \multimap B) \otimes (B \multimap C) \to A \multimap C$ corresponding by adjunction with the composite of the diagram

$$\begin{tikzcd}
((A \multimap B) \otimes (B \multimap C)) \otimes A \arrow[d, "\sigma_{A\multimap B, B\multimap C} \otimes 1_A"] \\
((B \multimap C) \otimes (A \multimap B)) \otimes A \arrow[d, "\alpha^{-1}_{B\multimap C, A\multimap B, A}"] \\
(B \multimap C) \otimes ((A \multimap B) \otimes A) \arrow[d, "1_{B\multimap C} \otimes \mathsf{ev}_{A,B}"] \\
(B \multimap C) \otimes B \arrow[d, "\mathsf{ev}_{B,C}"] \\
C;
\end{tikzcd}$$

- the canonical morphisms

$$\mathsf{u}_{A,B} = \Phi_{A,A\multimap B,B}(\mathsf{ev}_{A,B} \circ \sigma_{A,A\multimap B}) : A \to (A \multimap B) \multimap B.$$

Notice that these morphisms are all, by definition, natural in A, B and C.

Definition 2.6. A *symmetric monoidal category* is a category with symmetric tensor $\mathscr{C}$ equipped with an object $I \in \mathscr{C}$ called *unit* and an isomorphism

$$\lambda_A : I \otimes A \cong A$$

natural in A, such that the triangular diagram

$$\begin{tikzcd}
I \otimes (B \otimes C) \arrow[rr, "\alpha_{I,B,C}"] \arrow[dr, "\lambda_{B\otimes C}"'] & (T) & (I \otimes B) \otimes C \arrow[dl, "\lambda_B \otimes C"] \\
& B \otimes C &
\end{tikzcd}$$

commutes for all B, C in $\mathscr{C}$.

Definition 2.2 and its consequences extend to symmetric monoidal categories.

Remark 2.7. In any symmetric monoidal closed category, for any object A, the following diagram

$$\begin{tikzcd}
I \otimes A \arrow[rr, "\sigma_{I,A}"] & & A \otimes I \arrow[dl, "\rho_A"] \\
& A \arrow[ul, "\lambda_A^{-1}"] &
\end{tikzcd}$$

defines a unique canonical isomorphism $\rho_A = \sigma_{I,A} \circ \lambda_A^{-1} : A \otimes I \to A$, natural in A. By adjunction, it yields to an isomorphism $\tilde{\rho}_A : A \to I \multimap A$ natural in A.

Remark 2.8. There are various axiomatizations of a symmetric monoidal category in the literature. A proof of the fact that the axiomatization chosen above is equivalent to the usual ones (which include the ρ isomorphism, as the one in [Lan98], p.261) can be found in [HHS05], appendix A.

2.2 The Yoneda lemma

From now on, let $\mathscr{C}$ and $\mathscr{D}$ be small categories.

Definition 2.9. Consider a functor $F : \mathscr{C} \to \mathscr{D}$ and for every pair of objects A, B in $\mathscr{C}$, the mapping

$$\begin{array}{ccc} \mathsf{Hom}_{\mathscr{C}}(A,B) & \to & \mathsf{Hom}_{\mathscr{D}}(FA,FB) \\ f & \mapsto & Ff. \end{array}$$

1. The functor F is *faithful* when the above mapping is injective for all A, B.
2. The functor F is *full* when the above mapping is surjective for all A, B.
3. The functor F is *fully faithful* when the above mapping is bijective for all A, B.

Definition 2.10. A subcategory $\mathscr{D}$ of a category $\mathscr{C}$ consists of :

1. a subclass $|D|$ of the class of objects;
2. for every pair of objects C, C' of $|D|$, a subset $\mathsf{Hom}_{\mathscr{D}}(C,C') \subseteq \mathsf{Hom}_{\mathscr{C}}(C,C')$ such that
   - $f \in \mathsf{Hom}_{\mathscr{D}}(C,C')$ and $g \in \mathsf{Hom}_{\mathscr{D}}(C',C'') \Rightarrow g \circ f \in \mathsf{Hom}_{\mathscr{D}}(C,C'')$,
   - $\forall D \in \mathscr{D}, 1_D \in Hom_{\mathscr{D}}(D,D)$.

A subcategory $\mathscr{D}$ of $\mathscr{C}$ thus gives rise to a faithful inclusion functor from $\mathscr{D}$ to $\mathscr{C}$.

Definition 2.11. A subcategory $\mathscr{D}$ of a category $\mathscr{C}$ is called a *full subcategory* when the inclusion functor from $\mathscr{D}$ to $\mathscr{C}$ is also a full functor, i.e. when for all C, C' in $\mathscr{D}$, we have $\mathsf{Hom}_{\mathscr{D}}(C,C') = \mathsf{Hom}_{\mathscr{C}}(C,C')$.

Definition 2.12. Given an object $X \in \mathscr{C}$, we define a functor

$$\mathsf{Hom}_{\mathscr{C}}(X,-) : \mathscr{C} \to \mathsf{Set}$$

by putting

$$\mathsf{Hom}_{\mathscr{C}}(X,-)(A) = \mathsf{Hom}_{\mathscr{C}}(X,A)$$

for every object $A \in \mathscr{C}$ and

$$\begin{array}{cccc} \mathsf{Hom}_{\mathscr{C}}(X,f) : & \mathsf{Hom}_{\mathscr{C}}(X,A) & \to & \mathsf{Hom}_{\mathscr{C}}(X,B) \\ & g & \mapsto & f \circ g \end{array}$$

for every morphism $f : A \to B$ in $\mathscr{C}$.

We call such a functor the *covariant representable functor associated to* X and we denote it by h^X. We will say that a functor $F : \mathscr{C} \to \mathsf{Set}$ is *represented by an object* X in $\mathscr{C}$, if F is isomorphic to h^X in $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$. In particular h^X is represented by X.

Lemma 2.13. *In a symmetric monoidal closed category $\mathscr{C}$ we have the following isomorphism, natural in X and Y :*

$$h^X \circ H^Y \cong h^{X \otimes Y}. \tag{2}$$

Proof. By definitions of $h^{(-)}$ and $H^{(-)}$ and by isomorphism $\Phi^{-1}_{X,Y,Z}$ natural in all three variables, we have for every Z in $\mathscr{C}$:

$$\begin{aligned}(h^X \circ H^Y)(Z) &= \mathsf{Hom}_{\mathscr{C}}(X, Y \multimap Z) \\ &\cong \mathsf{Hom}_{\mathscr{C}}(X \otimes Y, Z) \\ &= h^{X \otimes Y}(Z).\end{aligned}$$

□

Definition 2.14. Consider a morphism $f : A \to B$ of $\mathscr{C}$. We obtain a natural transformation

$$\mathsf{Hom}(f, -) : \mathsf{Hom}_{\mathscr{C}}(A, -) \to \mathsf{Hom}_{\mathscr{C}}(B, -)$$

between the functors represented by A and B by putting for every object $C \in \mathscr{C}$ and every morphism $g \in \mathsf{Hom}_{\mathscr{C}}(A, C)$,

$$\mathsf{Hom}(f, -)_C(g) = \mathsf{Hom}(f, C) = g \circ f.$$

We denote $\mathsf{Hom}(f, -)$ by h^f.

Definition 2.15. The functor $\mathcal{Y} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ defined by

$$\mathcal{Y}(A) = h^A = \mathsf{Hom}_{\mathscr{C}}(A, -),$$

$$\mathcal{Y}(f) = h^f = \mathsf{Hom}(f, -),$$

for every object A in $\mathscr{C}$ and every morphism $f : A \to B$ of $\mathscr{C}$, is called the *contravariant Yoneda functor.*

Lemma 2.16 (The Yoneda lemma). *Let $F : \mathscr{C} \to \mathsf{Set}$ be a functor and $X \in \mathscr{C}$ an object. There is a canonical bijection*

$$\mathsf{Hom}_{\mathsf{Fun}(\mathscr{C},\mathsf{Set})}(F, h^X) \cong F(X)$$

natural in X and F.

Corollary 2.17. *The contravariant Yoneda functor $\mathcal{Y}$ is fully faithful.*

Remark 2.18. By "dualizing" definition 2.12 above, given any object X in $\mathscr{C}$ we can define the *contravariant representable functor associated to X*

$$h_X = \mathsf{Hom}_{\mathscr{C}}(-, X) : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}.$$

If $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$ denotes the category of contravariant functors from $\mathscr{C}$ to Set, the *covariant Yoneda functor* is the covariant functor

$$\mathcal{Y}_* : \mathscr{C} \to \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$$

defined by the formulas

$$\mathcal{Y}_*(X) = h_X = \mathsf{Hom}_{\mathscr{C}}(-, X)$$
$$\mathcal{Y}_*(f) = \mathsf{Hom}_{\mathscr{C}}(-, f)$$

for every object X and every morphism f of $\mathscr{C}$.

Moreover, we have a covariant version of the Yoneda lemma : there is a canonical bijection

$$\mathsf{Hom}_{\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})}(F, h_X) \cong F(X)$$

natural in $X \in \mathscr{C}$ and in $F \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$. It follows that the covariant Yoneda functor $\mathcal{Y}_*$, is also fully faithful.

Lemma 2.19. *In a symmetric monoidal closed category $\mathscr{C}$ we have an isomorphism*

$$\psi : H^{X \otimes Y} \to H^X \circ H^Y, \tag{3}$$

natural in X and Y in $\mathscr{C}$.

Proof. Using the associativity isomorphism α and the adjunction Φ, both natural in the three variables, we have for any objects A, Z in $\mathscr{C}$:

$$\begin{aligned}
\mathsf{Hom}_{\mathscr{C}}(A, H^{X \otimes Y}(Z)) &= \mathsf{Hom}_{\mathscr{C}}(A, (X \otimes Y) \multimap Z) \\
&\cong \mathsf{Hom}_{\mathscr{C}}(A \otimes (X \otimes Y), Z) \\
&\cong \mathsf{Hom}_{\mathscr{C}}((A \otimes X) \otimes Y, Z) \\
&\cong \mathsf{Hom}_{\mathscr{C}}(A \otimes X, Y \multimap Z) \\
&\cong \mathsf{Hom}_{\mathscr{C}}(A, X \multimap (Y \multimap Z)) \\
&= \mathsf{Hom}_{\mathscr{C}}(A, (H^X \circ H^Y)(Z)).
\end{aligned}$$

But $\mathsf{Hom}_{\mathscr{C}}(A, (X \otimes Y) \multimap Z)$ is $\mathcal{Y}_*((X \otimes Y) \multimap Z)$ applied to A and $\mathsf{Hom}_{\mathscr{C}}(A, X \multimap (Y \multimap Z))$ is $\mathcal{Y}_*(X \multimap (Y \multimap Z))$ applied to A. Since this is true for every A, using the fact that the covariant Yoneda functor is fully faithful, we then conclude that $(X \otimes Y) \multimap Z \cong X \multimap (Y \multimap Z)$, which is to say that for every Z in $\mathscr{C}$, $H^{X \otimes Y}(Z) \cong H^X \circ H^Y(Z)$. $\square$

For the missing proofs of this section and further details the user is invited to refer to [Bor94].

2.3 $*$-autonomous categories

Definition 2.20. Let $\mathscr{C}$ be a closed category with symmetric tensor. A functor $(-)^* : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ is a *duality functor* if there is an isomorphism $\mathsf{d}_{A,B} : A \multimap B \to B^* \multimap A^*$, natural in A and B, such that the diagram

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap C) \arrow[rr, "\mathsf{c}_{A,B,C}"] \arrow[d, "\mathsf{d}_{A,B} \otimes \mathsf{d}_{B,C}"'] & (D) & A \multimap C \arrow[d, "\mathsf{d}_{A,C}"] \\
(B^* \multimap A^*) \otimes (C^* \multimap B^*) \arrow[rr, "\mathsf{c}_{C^*,B^*,A^*} \circ \sigma_{B^* \multimap A^*, C^* \multimap B^*}"'] & & C^* \multimap A^*
\end{tikzcd}$$

commutes for all objects A, B and C in $\mathscr{C}$.

Definition 2.21. A *$*$-autonomous category* is a symmetric monoidal closed category equipped with a duality functor.

Definition 2.22. Let $\mathscr{C}$ be a closed category with symmetric tensor. An object $\perp$ is called *a dualizing object* if for every object A in $\mathscr{C}$ the canonical morphism $\mathsf{u}_{A,\perp} : A \to (A \multimap \perp) \multimap \perp$ (defined in remark 2.5) is an isomorphism.

Definition 2.23. A *$*$-autonomous category* is a symmetric monoidal closed category equipped with a dualizing object.

Proposition 2.24. *Definitions 2.21 and 2.23 of $*$-autonomous categories are equivalent.*

A proof of this proposition can be found in appendix A.

3 $*$-autonomous categories without units

3.1 Definition

By corollary 2.17, one can consider the category $\mathscr{C}^{\mathrm{op}}$ as a full subcategory of $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$. Thus, we can see objects of $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ as "generalized objects" of $\mathscr{C}$. Hence, we will call them virtual objects of $\mathscr{C}$, as the authors do in [LS06].

Definition 3.1. A *virtual object* of $\mathscr{C}$ is an object of $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$.

Remark 3.2. Let $\mathbb{A}$ be a virtual object of $\mathscr{C}$. If $\mathbb{A}$ is representable by an object A of $\mathscr{C}$, then by definition 2.12 we have $\mathbb{A} \simeq h^A = \mathsf{Hom}_{\mathscr{C}}(A, -)$.

Definition 3.3. Let $\mathscr{C}$ be a closed category with symmetric tensor. We define the *virtual tensor bifunctor* $(-) \otimes_v (-)$ to be :

$$\begin{array}{rccc} (-) \otimes_v (-) : & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \times \mathscr{C}^{\mathrm{op}} & \to & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \\ & (\mathbb{A}, X) & \mapsto & \mathbb{A} \circ H^X \end{array}$$

for every virtual object $\mathbb{A}$ of $\mathscr{C}$ and every object X of $\mathscr{C}$.

Remark 3.4. The application $(-) \otimes_v (-) : \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \times \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ is indeed functorial since it is obtained by composition of two functors as follows :

$$\begin{array}{ccccc} \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \times \mathscr{C}^{\mathrm{op}} & \xrightarrow{1 \times H^{(-)}} & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \times \mathsf{Fun}(\mathscr{C}, \mathscr{C}) & \xrightarrow{(-)\circ(-)} & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \\ (\mathbb{A}, X) & \mapsto & (\mathbb{A}, H^X) & \mapsto & \mathbb{A} \circ H^X \end{array}$$

where

- $1 \times H^{(-)}$ is the cartesian product of the identity functor $1_{\mathsf{Fun}(\mathscr{C}, \mathsf{Set})}$ with $H^{(-)}$ in Cat ;
- $(-) \circ (-)$ is the composition functor.

Given a closed category with symmetric tensor $\mathscr{C}$, we have defined a functor

$$\begin{array}{rccc} (-) \otimes_v (-) : & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \times \mathscr{C}^{\mathrm{op}} & \to & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \\ & (\mathbb{A}, X) & \mapsto & \mathbb{A} \circ H^X \end{array}$$

for every virtual object $\mathbb{A}$ of $\mathscr{C}$ and every object X of $\mathscr{C}$. If $\mathbb{A}$ is representable by an object A of $\mathscr{C}$, one can consider the functor obtained by precomposing $(-) \otimes_v (-)$ with the Yoneda functor $\mathcal{Y} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$:

$$\mathscr{C}^{\mathrm{op}} \times \mathscr{C}^{\mathrm{op}} \xrightarrow{\mathcal{Y} \times 1} \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \times \mathscr{C}^{\mathrm{op}} \xrightarrow{-\otimes_v -} \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$$

Thus, we obtain a functor :

$$\begin{array}{rccc} (- \otimes_v -)|_{\mathscr{C}^{\mathrm{op}}} : & \mathscr{C}^{\mathrm{op}} \times \mathscr{C}^{\mathrm{op}} & \to & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \\ & (A, X) & \mapsto & h^A \otimes_v X = h^A \circ H^X = \mathsf{Hom}_{\mathscr{C}}(A, X \multimap -) \end{array}$$

for all A, X in $\mathscr{C}$, and such that

$$\begin{array}{rccc} (s \otimes_v t)(Z) : & \mathsf{Hom}_{\mathscr{C}}(A, X \multimap Z) & \to & \mathsf{Hom}_{\mathscr{C}}(A', X' \multimap Z) \\ & h & \mapsto & (t \multimap 1_Z) \circ h \circ s \end{array}$$

for every morphism $s : A' \to A$, $t : X' \to X$ and object Z in $\mathscr{C}$.

On the other hand, if we compose the Yoneda functor $\mathcal{Y} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ with the usual tensor functor in $\mathscr{C}$ as follows

$$\mathscr{C}^{\mathrm{op}} \times \mathscr{C}^{\mathrm{op}} \xrightarrow{-\otimes-} \mathscr{C}^{\mathrm{op}} \xrightarrow{\mathcal{Y}} \mathsf{Fun}(\mathscr{C}, \mathsf{Set}),$$

we obtain the functor

$$\begin{array}{rccc} (- \otimes -)_{\mathcal{Y}} : & \mathscr{C}^{\mathrm{op}} \times \mathscr{C}^{\mathrm{op}} & \to & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \\ & (A, Y) & \mapsto & h^{A \otimes Y} = \mathsf{Hom}_{\mathscr{C}}(A \otimes Y, -) \end{array}$$

for all objects A, Y in $\mathscr{C}$, such that

$$\begin{array}{rccc} (f \otimes g)_{\mathcal{Y}}(Z) : & \mathsf{Hom}_{\mathscr{C}}(A \otimes Y, Z) & \to & \mathsf{Hom}_{\mathscr{C}}(A' \otimes Y', Z) \\ & k & \mapsto & k \circ (f \otimes g) \end{array}$$

for every morphism $f : A' \to A$, $g : Y' \to Y$ and object Z in $\mathscr{C}$.

Proposition 3.5. *Let $\mathscr{C}$ be a closed category with symmetric tensor. There is a canonical isomorphism between the functors $(- \otimes -)_{\mathcal{Y}}$ and $(- \otimes_v -)|_{\mathscr{C}^{\mathrm{op}}}$.*

Proof. We have a natural transformation

$$\Theta : (- \otimes -)_{\mathcal{Y}} \to (- \otimes_v -)|_{\mathscr{C}^{\mathrm{op}}}$$

given by the class of morphisms

$$\begin{array}{llcl} (\Theta_{A,X}(Z): & \mathsf{Hom}_{\mathscr{C}}(A \otimes X, Z) & \to & \mathsf{Hom}_{\mathscr{C}}(A, X \multimap Z))_{A,X \in \mathscr{C}^{\mathrm{op}}} \\ & h & \mapsto & \Phi_{A,X,Z}(h) \end{array}$$

where Φ is the adjunction isomorphism given in equation (1). These morphisms live in $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ since $\Phi_{A,X,Z}$ is natural in Z. Moreover since $\Phi_{A,X,Z}$ is also natural in A and X, the class of morphisms $(\Theta_{A,X})$ forms a natural transformation. □

Definition 3.6. A *symmetric monoidal closed category without units* is a closed category with symmetric tensor $\mathscr{C}$ together with a virtual object $\mathbb{I}$, and an isomorphism

$$\tilde{\lambda} : \mathbb{I} \otimes_v A \to h^A$$

natural in $A \in \mathscr{C}$ such that the diagram

$$\begin{tikzcd} \mathbb{I} \otimes_v (B \otimes C) \arrow[r, "\tilde{\alpha}_{\mathbb{I},B,C}"] \arrow[d, "\tilde{\lambda}_{B \otimes C}"'] \arrow[dr, phantom, "(Tv)"] & (\mathbb{I} \otimes_v B) \otimes_v C \arrow[d, "\tilde{\lambda}_{B \otimes_v C}"] \\ h^{B \otimes C} \arrow[r, "\Theta_{B,C}"'] & h^B \otimes_v C \end{tikzcd}$$

commutes for all objects B, C in $\mathscr{C}$, where

$$\tilde{\alpha}_{\mathbb{I},B,C}(Z) = \mathbb{I}(\psi_{B,C}(Z))$$

for all B, C, Z in $\mathscr{C}$ (see lemma 2.19 for the definition of ψ).

We shall say that $\mathbb{I}$ is a *virtual unit* of $\mathscr{C}$.

Remark 3.7. Let us unfold the above definition of

$$\tilde{\alpha}_{\mathbb{I},B,C}(Z) = \mathbb{I}(\psi_{B,C}(Z)) : (\mathbb{I} \otimes_v (B \otimes C))(Z) \to ((\mathbb{I} \otimes_v B) \otimes_v C)(Z).$$

We have :

- by definitions of the virtual tensor and the $H^{(-)}$ functors

$$(\mathbb{I} \otimes_v (B \otimes C))(Z) = (\mathbb{I} \circ H^{B\otimes C})(Z) = \mathbb{I}((B \otimes C) \multimap Z)$$

  and

$$\begin{array}{rcl} ((\mathbb{I} \otimes_v B) \otimes_v C)(Z) & = & ((\mathbb{I} \otimes_v B) \circ H^C)(Z) \\ & = & (\mathbb{I} \otimes_v B)(C \multimap Z) \\ & = & (\mathbb{I} \circ H^B)(C \multimap Z) \\ & = & \mathbb{I}(B \multimap (C \multimap Z)); \end{array}$$

- by lemma 2.19 an isomorphism for every Z in $\mathscr{C}$

$$\psi_{B,C}(Z) : (B \otimes C) \multimap Z \cong (B \multimap C) \multimap Z$$

  natural in B and C.

Thus, ψ being an isomorphism and $\mathbb{I}$ being a functor, $\tilde{\alpha}_{\mathbb{I},B,C} = \mathbb{I}(\psi_{B,C})$ is an isomorphism from $\mathbb{I}\otimes_v(B\otimes C)$ to $(\mathbb{I}\otimes_v B)\otimes_v C$. Moreover, $\mathbb{I}$ being an object of the category $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ (where morphisms are natural transformations), $\tilde{\alpha}_{\mathbb{I},B,C}$ is natural in its first variable.

Definition 3.8. A $*$-*autonomous category without units* is a symmetric monoidal closed category without units equipped with a duality functor.

This axiomatization of $*$-autonomous categories without units is closely related, though this is not blindingly obvious, to the one of $*$-autonomous categories. Diagram (Tv) for $*$-autonomous categories without units clearly mimics diagram (T) of $*$-autonomous ones (modulo the natural transformation Θ between the usual tensor and the virtual one). Isomorphism $\tilde{\alpha}$ of $*$-autonomous categories without units is defined using isomorphism ψ which is himself obtained via the isomorphism α of $*$-autonomous categories. The duality functor is the same in both axiomatizations. These similarities are formalized in the next section.

3.2 Correspondence

Lemma 3.9. *Let $\mathscr{C}$ be a $*$-autonomous category and I its unit object. For all A, B in $\mathscr{C}$ the following diagram*

$$\begin{tikzcd} h^I \otimes_v (A \otimes B) \arrow[r, "\Theta^{-1}_{I,A\otimes B}"] \arrow[d, "\tilde{\alpha}_{h^I,A,B}"'] & h^{I\otimes(A\otimes B)} \arrow[d, "h^{\alpha_{I,A,B}}"] \\ (h^I \otimes_v A) \otimes_v B \arrow[r, "\Theta^{-1}_{I\otimes A,B}\circ(\Theta^{-1}_{I,A}\otimes_v 1_B)"'] & h^{(I\otimes A)\otimes B} \end{tikzcd}$$

commutes.

Proof. The proof is an unfolding of the definitions of the virtual tensor and $\tilde{\alpha}$. First, let us apply to the above diagram a $Z \in \mathscr{C}$. We get:

$$\begin{tikzcd}
\mathsf{Hom}(I, (A \otimes B) \multimap Z) \arrow[r, "\Phi^{-1}_{I,A\otimes B,Z}"] \arrow[d, "\tilde{\alpha}_{h^I,A,B}(Z)"'] & \mathsf{Hom}(I \otimes (A \otimes B), Z) \arrow[d, "h^{\alpha_{I,A,B}}(Z)"] \\
\mathsf{Hom}(I, A \multimap (B \multimap Z)) \arrow[r, "\Phi^{-1}_{I\otimes A,B,Z} \circ \Phi^{-1}_{I,A,B\multimap Z}"] & \mathsf{Hom}((I \otimes A) \otimes B, Z).
\end{tikzcd}$$

Now, $\tilde{\alpha}_{h^I,A,B}(Z)$ is by definition $h^I(\psi_{A,B}(Z))$ which is defined in the proof of lemma 2.19 by the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Hom}(I, (A \otimes B) \multimap Z) \arrow[r, "\Phi^{-1}_{I,A\otimes B,Z}"] \arrow[d, "\tilde{\alpha}_{h^I,A,B}(Z)"'] & \mathsf{Hom}(I \otimes (A \otimes B), Z) \arrow[d, "h^{\alpha_{I,A,B}}(Z)"] \\
\mathsf{Hom}(I, A \multimap (B \multimap Z)) & \mathsf{Hom}((I \otimes A) \otimes B, Z). \arrow[l, "\Phi_{I,A,B\multimap Z} \circ \Phi_{I\otimes A,B,Z}"]
\end{tikzcd}$$

By replacing $\tilde{\alpha}_{h^I,A,B}(Z)$ by this definition in the first diagram, one can check that it commutes.

$\square$

Proposition 3.10. *1. If $\mathscr{C}$ is a $*$-autonomous category without units such that its virtual unit $\mathbb{I}$ is representable by an object I of $\mathscr{C}$, then $\mathscr{C}$ is a $*$-autonomous category with I being its unit object.*

2. If $\mathscr{C}$ is a $*$-autonomous category with unit object I, then $\mathscr{C}$ is a $*$-autonomous category without units where h^I is its virtual unit.

Proof. **Proof of 1**:

since $\mathscr{C}$ is $*$-autonomous without units, $\mathscr{C}$ is by definition a closed category with symmetric tensor. Therefore $\mathscr{C}$ is already equipped with isomorphisms α and σ such that the hexagonal and pentagonal diagrams commute.

It remains to show that the object I which represents $\mathbb{I}$ is the unit object of $\mathscr{C}$ and that the triangular diagram (T) commutes.

Let $i : \mathbb{I} \to h^I$ be the isomorphism by which I represents $\mathbb{I}$. The commuting diagram

$$\begin{tikzcd}
h^{I\otimes A} \arrow[r, "\Theta_{I,A}"] \arrow[dr, "h^{\lambda_A}"'] & h^I \otimes_v A \arrow[r, "i^{-1}\otimes_v 1_A"] & \mathbb{I} \otimes_v A \arrow[dl, "\tilde{\lambda}_A"] \\
& h^A &
\end{tikzcd}$$

defines a unique canonical isomorphism

$$h^{\lambda_A} = \tilde{\lambda}_A \circ (i^{-1} \otimes_v 1_A) \circ \Theta_{(I,A)} : I \otimes A \to A \tag{4}$$

natural in A in $\mathscr{C}$.

We will use the following diagram to show that diagram (h^T) commutes :

$$\begin{tikzcd}
\mathbb{I} \otimes_v (A \otimes B) \arrow[rrr, "\Theta^{-1}_{I,A\otimes B}\circ(i\otimes_v 1_{A\otimes B})"] \arrow[dr, "\tilde{\lambda}_{A\otimes B}"'] \arrow[ddd, "\tilde{\alpha}_{\mathbb{I},A,B}"] & & & h^{I\otimes(A\otimes B)} \arrow[dl, "h^{\lambda_{A\otimes B}}"] \arrow[ddd, "h^{\alpha_{I,A,B}}"] \\
& h^{A\otimes B} \arrow[r, equal] \arrow[d, "\Theta_{A,B}"] \arrow[ur, phantom, "(1)"] & h^{A\otimes B} \arrow[d, equal] & \\
\arrow[r, phantom, "(Tv)"] & h^{A} \otimes_v B \arrow[r, "\Theta^{-1}_{A,B}"'] \arrow[ur, phantom, "(2)"] & h^{A\otimes B} \arrow[r, phantom, "(h^T)"] & {} \\
(\mathbb{I} \otimes_v A) \otimes_v B \arrow[uur, "\tilde{\lambda}_A\otimes_v 1_B"] \arrow[rrr, "\Theta^{-1}_{I\otimes A,B}\circ(\Theta^{-1}_{I,A}\otimes_v 1_B)\circ((i\otimes_v 1_A)\otimes_v 1_B)"'] & & & h^{(I\otimes A)\otimes B} \arrow[ul, "h^{\lambda_A\otimes 1_B}"'] \arrow[ull, phantom, "(3)"]
\end{tikzcd}$$

Diagram 3.10

- Diagram (Tv) commutes by hypothesis.
- Diagram (2) commutes since Θ is an isomorphism.
- Diagram (3) decomposes as the diagram:

$$\begin{tikzcd}
(\mathbb{I} \otimes_v A) \otimes_v B \arrow[r, "(i\otimes_v 1_A)\otimes_v 1_B"] \arrow[dr, "\tilde{\lambda}_A\otimes_v 1_B"'] & (h^I \otimes_v A) \otimes_v B \arrow[r, "\Theta^{-1}_{I,A}\otimes_v 1_B"] \arrow[d, phantom, "(3_a)"] & h^{I\otimes A} \otimes_v B \arrow[r, "\Theta^{-1}_{I\otimes A,B}"] \arrow[dl, "h^{\lambda_A}\otimes_v 1_B"'] \arrow[d, phantom, "(3_b)"] & h^{(I\otimes A)\otimes B} \arrow[dl, "h^{\lambda_A\otimes 1_B}"] \\
& h^{A} \otimes_v B \arrow[r, "\Theta^{-1}_{A,B}"] & h^{A\otimes B} &
\end{tikzcd}$$

  One can easily check by replacing h^{λ_A} by its definition (equation (4) above) that diagram (3_a) commutes.

  Diagram (3_b) is a diagram of naturality of Θ^{-1} in its first variable.

  Thus, diagram (3) commutes.

- One can easily check that diagram (1) commutes by definition of $h^{\lambda_{A\otimes B}}$ (equation (4) above).

- The outer diagram decomposes as :

$$\begin{tikzcd}
\mathbb{I} \otimes_v (A \otimes B) \arrow[r, "i\otimes_v 1_{A\otimes B}"] \arrow[d, "\tilde{\alpha}_{\mathbb{I},A,B}"'] & h^I \otimes_v (A \otimes B) \arrow[r, "\Theta^{-1}_{I,A\otimes B}"] \arrow[d, "\tilde{\alpha}_{h^I,A,B}"] & h^{I\otimes(A\otimes B)} \arrow[d, "h^{\alpha_{I,A,B}}"] \\
(\mathbb{I} \otimes_v A) \otimes_v B \arrow[r, "(i\otimes_v 1_A)\otimes_v 1_B"'] & (h^I \otimes_v A) \otimes_v B \arrow[r, "\Theta^{-1}_{I\otimes A,B}\circ(\Theta^{-1}_{I,A}\otimes_v 1_B)"'] & h^{(I\otimes A)\otimes B}.
\end{tikzcd}$$

  The left square is a naturality diagram of $\tilde{\alpha}$ in its first variable (see remark 3.7) and hence commutes.

  The right square commutes by proposition 3.9.

  Hence the outer rectangle diagram commutes.

- Hence, diagram (h^T) commutes. By the contravariant Yoneda lemma, it follows that diagram (T) commutes.

So far, we have proved that $\mathscr{C}$ is a symmetric monoidal closed category. Since $\mathscr{C}$ is by hypothesis equipped with a duality functor, $\mathscr{C}$ is also $*$-autonomous. By proposition 2.24 and its proof (see appendix A), $\mathscr{C}$ is equipped with a dualizing object $\perp = I^*$.

Proof of 2 :

since $\mathscr{C}$ is $*$-autonomous, $\mathscr{C}$ is a closed category with symmetric tensor. Let $h^I \in \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ be the functor representing the unit I of $\mathscr{C}$. The commuting diagram

$$\begin{tikzcd}
h^{I\otimes A} \arrow[dr, "h^{\lambda_A}"'] & h^I \otimes_v A \arrow[l, "\Theta^{-1}_{I,A}"'] \arrow[d, "\tilde{\lambda}_A"] \\
 & h^A
\end{tikzcd}$$

defines a unique canonical isomorphism

$$\tilde{\lambda}_A = h^{\lambda_A} \circ \Theta^{-1}_{I,A} : h^I \otimes_v A \to h^A \tag{5}$$

natural in A.

We define, for any Z in $\mathscr{C}$, $\tilde{\alpha}_{h^I,B,C}(Z) = h^I(\psi_{B,C}(Z))$, which is a unique isomorphism from $h^I \otimes_v (B \otimes C))$ to $(h^I \otimes_v B) \otimes_v C$ (see remark 3.7).

Once again in this proof we use diagram 3.10 above, where we replace everywhere $\mathbb{I}$ by h^I and where isomorphism i is now the identity morphism. But this time we use that diagram to prove that diagram (Tv) commutes, knowing that diagram (h^T) commutes (because diagram (T) commutes by hypothesis and by using the contravariant Yoneda lemma). The other diagrams involved commute for the same reasons as they did above. Hence, diagram (Tv), commutes. Since $\mathscr{C}$ is already equipped with a duality functor, we conclude that $\mathscr{C}$ is a $*$-autonomous category without units with h^I being a virtual unit for it. $\square$

4 Unit-free multiplicative linear logic (MLL^-)

4.1 Sequent calculus for MLL^-

The set of formulas of MLL^- is defined inductively as

$$\mathscr{F} := \mathscr{A} \mid \mathscr{A}^* \mid \mathscr{F} \invamp \mathscr{F} \mid \mathscr{F} \otimes \mathscr{F}$$

where $\mathscr{A} = \{a, b, c, \ldots\}$ is a countable set of propositional variables, and $\mathscr{A}^* = \{a^*, b^*, c^*, \ldots\}$ are their duals. In the following, we will call *atoms* the elements of $\mathscr{A} \cup \mathscr{A}^*$.

The dual of a formula is defined inductively as

$$a^{**} = a \qquad (A \otimes B)^* = B^* \invamp A^* \qquad (A \invamp B)^* = B^* \otimes A^*.$$

The linear implication is defined as $A \multimap B := A^* \invamp B$.

Here is the set of inference rules for MLL^- given in the formalism of the sequent calculus:

$$\begin{array}{ccc} id\,\dfrac{}{\vdash A^*, A} & exch\,\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta} & \invamp\,\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \invamp B, \Delta} \\ \otimes\,\dfrac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} & & cut\,\dfrac{\vdash \Gamma, A \quad \vdash A^*, \Delta}{\vdash \Gamma, \Delta} \end{array} \tag{6}$$

4.2 Proof-nets for MLL^-

Proofs presented using sequent calculus contain a lot of details that are sometimes uninteresting: consider for example how many uninterestingly different ways there are to form a proof of $\vdash \Gamma, (A_1 \invamp A_2), \ldots, (A_{n-1} \invamp A_n)$ from a derivation of $\vdash \Gamma, A_1, A_2, \ldots, A_n$.

The theory of proof-nets for MLL^- was introduced by Girard in [Gir87]. A proof-net is a formalism for representing proofs in such a way that it identifies proofs that differ because of inessential or useless permutation of rules.

In this paper we consider the coherence graph based proof-nets for MLL^-, where the proof-net is obtained by drawing the coherence graph through the sequent calculus proof and where the essence of a proof is captured by the axiom links. For the definition and further details on the coherence graph based proof-nets the reader can refer to [Str06] and [Hug05].

These proof-nets form a category whose objects are the formulas built from $\mathscr{A} \cup \mathscr{A}^*$ and whose arrows are the cut-free proof-nets. More precisely, for every formula A there is an identity proof-net of $\vdash A^*, A$ with an axiom link between A^* and A. Moreover, for any two proof-nets f of $\vdash A^*, B$ and g of $\vdash B^*, C$, the composition $g \circ f$ is the result of applying the cut-elimination procedure to $\vdash A^*, B \oplus B^*, C$, where $\oplus$ is the cut connective. That this is well-defined and associative follows from the strong normalization of the cut elimination procedure.

We denote this category by $\mathsf{PN}^-(\mathscr{A})$.

4.3 The category of MLL^- proof-nets

Proposition 4.1. *For every set of propositional variables $\mathscr{A}$, the category $\mathsf{PN}^-(\mathscr{A})$ is a $*$-autonomous category without units.*

Proof. Sketch: The main idea here is that $\mathbb{I}(A)$ is the set of all proof-nets of $\vdash A$, i.e. proofs of an MLL^- formula A from nothing. Hence, the isomorphism

$$\tilde{\lambda}_A(B) : \mathbb{I}(A \multimap B) \to \mathsf{Hom}(A, B)$$

says that the set of proof-nets of $\vdash A \multimap B$ is the set of maps $A \to B$.

Everything else in $\mathsf{PN}^-(\mathscr{A})$, (as long as the units do not interfere) behaves as in the category of cut-free proof-nets for MLL, which is a $*$-autonomous category as showed in [LS06]. More precisely, we have :

- the bifunctor $- \otimes - : \mathsf{PN}^-(\mathscr{A}) \times \mathsf{PN}^-(\mathscr{A}) \to \mathsf{PN}^-(\mathscr{A})$ is given by the operation $\otimes$ on formulas and for any two arrows $f : A \to B$ proof-net of $\vdash A^*, B$ and $g : C \to D$ proof-net of $\vdash C^*, D$, there is a uniquely defined arrow $f \otimes G : A \otimes B \to C \otimes D$ corresponding to a proof-net of $\vdash B^* \parr A^*, C \otimes D$;

- a proof-net of
$$\vdash (C^* \parr B^*) \parr A^*, (A \otimes B) \otimes C$$
corresponding to the isomorphism
$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C;$$

- a proof-net of
$$\vdash B^* \parr A^*, A \otimes B$$
corresponding to the isomorphism
$$\sigma_{A,B} : A \otimes B \to B \otimes A;$$

- the duality functor $(-)^*$ is defined on the formulas as in section 4.1 above and on arrows by assigning to an arrow $f : A \to B$, proof-net of $\vdash A^*, B$, the arrow $f^* : B^* \to A^*$ which is a proof-net of $\vdash B, A^*$;

- there is a natural bijection
$$\begin{array}{ccc} \mathsf{Hom}(A \otimes B, C) & \cong & \mathsf{Hom}(A, B^* \parr C) \\ \vdash B^* \parr A^*, C & \to & \vdash A^*, B^* \parr C. \end{array}$$

$\square$

5 Free $*$-autonomous categories without units

5.1 Definition

Definition 5.1. Let $\mathscr{C}$ be a category, $U : \mathscr{C} \to \mathsf{Set}$ a functor and S a set. The *free object on the set* S is an object $c \in \mathscr{C}$ together with an application $S \to Uc$ such that for any object $d \in \mathscr{C}$ together with an application $S \to Ud$, there is one and only one morphism $f : c \to d$ in $\mathscr{C}$ which makes the diagram

$$\begin{tikzcd} & Uc \arrow[dd, "Uf"] \\ S \arrow[ur] \arrow[dr] & \\ & Ud \end{tikzcd}$$

commute.

In particular, we have the following definition. Let Obj be the functor which associates to a $*$-autonomous category without units $\mathscr{C}$, the set $\mathsf{Obj}(\mathscr{C})$ of its objects.

Definition 5.2. Let S be a set. The free $*$-autonomous category without units on S is a $*$-autonomous category without units $\mathscr{C}$ together with an application $S \to \mathsf{Obj}(\mathscr{C})$ such that for every $*$-autonomous without units category $\mathscr{D}$ and every application $S \to \mathsf{Obj}(\mathscr{D})$, there is one and only one strict $*$-autonomous functor without units $F : \mathscr{C} \to \mathscr{D}$ such that the diagram

$$\begin{tikzcd} & \mathsf{Obj}(\mathscr{C}) \arrow[dd, "\mathsf{Obj}(F)"] \\ \mathscr{A} \arrow[ur, "\nu_{\mathscr{A}}"] \arrow[dr] & \\ & \mathsf{Obj}(\mathscr{D}) \end{tikzcd}$$

commutes.

We want the category $\mathsf{PN}^-(\mathscr{A})$ to be the free $*$-autonomous category without units on the set $\mathscr{A}$. For this, according to the definition above a notion of $*$-autonomous without units functor is needed.

5.2 $*$-autonomous functors

Definition 5.3. Let $\mathscr{C}$ and $\mathscr{C}'$ be two $*$-autonomous categories with units I and I' respectively. A functor $F : \mathscr{C} \to \mathscr{C}'$ is said to be *strict $*$-autonomous* if we have :

1. $F(A \otimes B) = F(A) \otimes F(B)$ for every object A, B in $\mathscr{C}$;

2. $F(I) = I'$;
3. $F(A \multimap B) = FA \multimap FB$ for every object A, B in $\mathscr{C}$;
4. $F(\perp) = \perp'$ where $\perp$ and $\perp'$ are the dualizing objects of $\mathscr{C}$ and $\mathscr{C}'$ respectively.

In a similar manner, we define a notion of strict $*$-autonomous without units functor.

Definition 5.4. Let $\mathscr{C}$ and $\mathscr{C}'$ be two $*$-autonomous categories without units, with virtual units $\mathbb{I}$ and $\mathbb{I}'$ respectively. A functor $F : \mathscr{C} \to \mathscr{C}'$ is said to be *strict $*$-autonomous without units* if we have :

- $F(A \otimes B) = F(A) \otimes F(B)$ for every object A, B in $\mathscr{C}$;
- $F(A \multimap B) = FA \multimap FB$ for every object A, B in $\mathscr{C}$;
- there is a natural isomorphism $\beta : \mathbb{I}' \circ F \to \mathbb{I}$.

The natural transformation β above is required for such a functor to preserve the virtual units. Notice that this definition of β is the only obvious way for F to preserve the virtual units.

5.3 The MLL^- case

For every set of propositional variables $\mathscr{A}$, the category $\mathsf{PN}^-(\mathscr{A})$ is not the free $*$-autonomous category without units on $\mathscr{A}$.

To see why, let us consider the terminal category $(*)$ (i.e. the one-morphism category which is obviously a $*$-autonomous category without units) and let h^* be its virtual unit (with $*$ being its only object). There should exist a $*$-autonomous functor without units $F : \mathsf{PN}^-(\mathscr{A}) \to (*)$.

Let A be an object of $\mathsf{PN}^-(\mathscr{A})$, i.e. a formula built on the set $\mathscr{A} \cup \mathscr{A}^*$. By definition of a strict $*$-autonomous functor without units (see 5.4) there should exist a natural isomorphism

$$\beta : (h^* \circ F)(A) \to \mathbb{I}(A). \tag{7}$$

Now, F sends A to the only object $*$, hence $(h^* \circ F)(A) = h^*(*) = Hom_{(*)}(*, *)$ which has exactly one element: 1_*. But on the other side $\mathbb{I}(A)$ is the set of all proof-nets of $\vdash A$. Hence isomorphism (7) above requires that for any formula A, there is exactly one proof-net of $\vdash A$, which is obviously false.

6 Conclusion and future work

We have formalized the definition of $*$-autonomous categories without units presented in [LS05] and proved some fundamental properties of it.

Although the category $\mathsf{PN}^-(\mathscr{A})$ of cut-free proof-nets for MLL^- is such a $*$-autonomous category without units, we have proved that $\mathsf{PN}^-(\mathscr{A})$ is not the free $*$-autonomous category without units on the set of variables $\mathscr{A}$.

More precisely, the problem seems to lie in the definition of a strict $*$-autonomous functor without units. To be even more precise, if F is such a functor between two $*$-autonomous categories without units, one has to find the right way (if there is one) for F to preserve the virtual units of these categories.

7 Appendix A

In this appendix we give a proof of proposition 2.24 which states that definitions 2.21 and 2.23 of $*$-autonomous categories are equivalent.

Proof. First we prove that definition 2.21 implies definition 2.23. Suppose we have a symmetric monoidal closed category $\mathscr{C}$ and a duality functor $(-)^* : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ such that diagram (D) commutes. We denote I^* by $\perp$. First, let us check the commutativity of the following diagram

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap \perp) \arrow[r] \arrow[d, "\mathsf{d}_{A,B} \otimes \mathsf{d}_{B,\perp}"'] \arrow[dr, phantom, "(a)"] & A \multimap \perp \arrow[d, "\mathsf{d}_{A,\perp}"] \\
(B^* \multimap A^*) \otimes (I \multimap B^*) \arrow[r] \arrow[d, "1 \otimes \tilde{\rho}^{-1}_{B^*}"'] \arrow[dr, phantom, "(b)"] & I \multimap A^* \arrow[d, "\tilde{\rho}^{-1}_{A^*}"] \\
(B^* \multimap A^*) \otimes B^* \arrow[r] & A^*
\end{tikzcd}$$

in which the horizontal arrows are instances of the composition morphism c.

Diagram (a) is diagram (D) specialized to the case $C = \perp$, hence commutes by hypothesis. Diagram (b) is a naturality diagram of the natural transformation

$$(B^* \multimap A^*) \otimes (I \multimap -) \to (I \multimap -).$$

Hence diagram (b) commutes. We conclude that the hole diagram above commutes. This is to say that

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap \perp) \arrow[r] \arrow[d, "\mathsf{d}_{A,B} \otimes \mathsf{s}_B"'] \arrow[dr, phantom, "(ab)"] & A \multimap \perp \arrow[d, "\mathsf{s}_A"] \\
(B^* \multimap A^*) \otimes B^* \arrow[r] & A^*
\end{tikzcd}$$

commutes, with $\mathsf{s}_A = \tilde{\rho}^{-1}_{A^*} \circ \mathsf{d}_{A,\perp} : A \multimap \perp \to A^*$.

Now, remark 2.4 says that the adjoint diagram of diagram (ab) also commutes (the reader should notice that this is true here only because the maps $\mathsf{d}_{A,B}$ and s_B are isomorphisms). Specialized to the case $A = I$ we get the commuting diagram

$$\begin{tikzcd}
I \multimap B \arrow[rr] \arrow[d, "\mathsf{d}_{I,B}"'] \arrow[drr, phantom, "(\tilde{ab})"] & & (B \multimap \perp) \multimap (I \multimap \perp) \arrow[d, "\mathsf{s}_B^{-1} \multimap \mathsf{s}_I"] \\
B^* \multimap \perp \arrow[rr, equal] & & B^* \multimap \perp\ .
\end{tikzcd}$$

Now, consider the diagram

$$\begin{tikzcd}
B \arrow[rr, "\mathsf{u}_{B,\perp}"] \arrow[d, "\tilde{\rho}_B"'] \arrow[drr, phantom, "(c)"] & & (B \multimap \perp) \multimap \perp \arrow[dd, "1_{B \multimap \perp} \multimap \tilde{\rho}_\perp"] \\
I \multimap B \arrow[d, "\mathsf{d}_{B,\perp}"'] \arrow[drr, "\phi(\mathsf{c}_{I,B,\perp})"] \arrow[dr, phantom, "(\tilde{ab})"] & & \\
B^* \multimap \perp & & (B \multimap \perp) \multimap (I \multimap \perp). \arrow[ll, "\mathsf{s}_B^{-1} \multimap \mathsf{s}_I"']
\end{tikzcd}$$

Diagram (c) is a naturality diagram of the natural transformation

$$I \multimap - \to (B \multimap \perp) \multimap (I \multimap -).$$

Hence diagram (c) commutes. Since we just proved that diagram $(\tilde{ab})$ also commutes, the outer diagram

$$\begin{tikzcd}
B \arrow[r, "\mathsf{u}_{B,\perp}"] \arrow[d, "\mathsf{d}_{B,\perp} \circ \tilde{\rho}_B"'] & (B \multimap \perp) \multimap \perp \arrow[d, "1_{B \multimap \perp} \multimap \tilde{\rho}_\perp"] \\
B^* \multimap \perp & (B \multimap \perp) \multimap (I \multimap \perp) \arrow[l, "\mathsf{s}_B^{-1} \multimap \mathsf{s}_I"]
\end{tikzcd}$$

commutes. This is to say that diagram

$$\begin{tikzcd}
B \arrow[rr, "\mathsf{u}_{B,\perp}"] \arrow[dr, "\mathsf{d}_{B,\perp} \circ \tilde{\rho}_B"'] & & (B \multimap \perp) \multimap \perp \arrow[dl, "(\mathsf{s}_B^{-1} \multimap \mathsf{s}_I) \circ (1_{B \multimap \perp} \multimap \tilde{\rho}_\perp)"] \\
& B^* \multimap \perp &
\end{tikzcd}$$

commutes. But both diagonal maps are isomorphisms and hence so is the horizontal map, which shows that definition 2.23 is satisfied.

Now, let us prove that definition 2.23 implies definition 2.21. If $\mathscr{C}$ equipped with a dualizing object $\perp$ satisfies definition 2.23, we define for all A in $\mathscr{C}$, $A^* = A \multimap \perp$. With this definition of the duality functor, using isomorphism ψ defined in lemma (2.19), we define $\mathsf{d}_{A,B} : A \multimap B \to B^* \multimap A^*$ to be the composite of the diagram

$$\begin{array}{c}
A \multimap B \\
\Big\downarrow {\scriptstyle 1_A \multimap \mathsf{u}^{-1}_{B,\perp}} \\
A \multimap ((B \multimap \perp) \multimap \perp) \\
\Big\downarrow {\scriptstyle \psi_{A,B\multimap\perp,\perp}} \\
(A \otimes (B \multimap \perp)) \multimap \perp \\
\Big\downarrow {\scriptstyle \sigma_{A,B\multimap\perp} \multimap 1_\perp} \\
((B \multimap \perp) \otimes A) \multimap \perp \\
\Big\downarrow {\scriptstyle \psi_{B\multimap\perp,A,\perp}} \\
(B \multimap \perp) \multimap (A \multimap \perp) = B^* \multimap A^*.
\end{array}$$

Since every arrow of the above diagram is an isomorphism, so is $\mathsf{d}_{A,B}$. Moreover, since these arrows are all natural in A and B, so is $\mathsf{d}_{A,B}$.

Finally, diagram (D) :

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap C) \arrow[r, "\mathsf{c}_{A,B,C}"] \arrow[d, "\mathsf{d}_{A,B} \otimes \mathsf{d}_{B,C}"'] \arrow[dr, phantom, "(D)"] & A \multimap C \arrow[d, "\mathsf{d}_{A,C}"] \\
(B^* \multimap A^*) \otimes (C^* \multimap B^*) \arrow[r, "\mathsf{c}_{C^*,B^*,A^*} \circ \sigma_{B^*\multimap A^*, C^*\multimap B^*}"'] & C^* \multimap A^*
\end{tikzcd}$$

commutes. To see why the reader is invited to refer to [KO98]. In this paper the authors introduce a family of type theories as internal language for symmetric monoidal closed categories. They prove that there is a canonical interpretation of the type theories in symmetric monoidal closed categories which is complete. This yields in particular to a coherence theorem for such categories (Theorem 5.11, page 54). According to this theorem, since $((A \multimap B) \otimes (B \multimap C)) \multimap (C^* \multimap A^*)$ verifies the hypothesis, there is at most one map from $(A \multimap B) \otimes (B \multimap C)$ to $C^* \multimap A^*$. Hence diagram (D) commutes. $\square$

8 Appendix B : Résumé

8.1 Catégories avec produit tensoriel

Definition 8.1. Une *catégorie avec un produit tensoriel symétrique* est une catégorie $\mathscr{C}$ munie d'un bifoncteur $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ et d'isomorphismes

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C,$$
$$\sigma_{A,B} : A \otimes B \cong B \otimes A,$$

naturels en A, B, C dans $\mathscr{C}$, tels que :

1. on a $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A\otimes B}$ pour tout A, B dans $\mathscr{C}$;
2. le pentagone

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \arrow[d, "\alpha\otimes 1"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"] & & (A \otimes (B \otimes C)) \otimes D
\end{tikzcd}$$

   est commutatif pour tout A, B, C, D dans $\mathscr{C}$;
3. l'hexagone

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "1\otimes\sigma"'] & (A \otimes B) \otimes C \arrow[r, "\sigma"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha"] & (A \otimes C) \otimes B \arrow[r, "\sigma\otimes 1"] & (C \otimes A) \otimes B
\end{tikzcd}$$

   est commutatif pour tout A, B, C dans $\mathscr{C}$.

Definition 8.2. Une catégorie avec produit tensoriel symétrique $\mathscr{C}$ est dite *fermée* si pour tout objet B de $\mathscr{C}$, le foncteur $(-) \otimes B$ admet un adjoint à droite.

Nous noterons $B \multimap (-)$ l'adjoint à droite du foncteur $(-) \otimes B$. Par adjonction, on a un isomorphisme

$$\Phi_{A,B,C} : \mathsf{Hom}_{\mathscr{C}}(A \otimes B, C) \cong \mathsf{Hom}_{\mathscr{C}}(A, B \multimap C) \tag{8}$$

naturel en A et C dans $\mathscr{C}$. On obtient ainsi un bifoncteur $(-) \multimap (-)$ de $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ vers C. De plus, ϕ est naturel en A, B et C. Ces données permettent de définir un isomorphisme

$$\psi_{A,B,C} : (A \otimes B) \multimap C \cong A \multimap (B \multimap C) \tag{9}$$

naturel en A, B et C dans $\mathscr{C}$.

Definition 8.3. Une *catégorie monoïdale symétrique* est une catégorie avec un produit tensoriel symétrique $\mathscr{C}$ munie d'un objet I de $\mathscr{C}$ appelé l'*unité* et d'un isomorphisme

$$\lambda_A : I \otimes A \cong A$$

naturel en A, tels que le triangle

$$\begin{tikzcd}
I \otimes (B \otimes C) \arrow[rr, "\alpha_{I,B,C}"] \arrow[dr, "\lambda_{B\otimes C}"'] & \arrow[d, phantom, "(T)"] & (I \otimes B) \otimes C \arrow[dl, "\lambda_B \otimes C"] \\
& B \otimes C &
\end{tikzcd}$$

soit commutatif pour tout B, C dans $\mathscr{C}$.

8.2 Catégories $*$-autonomes

Definition 8.4. Soit $\mathscr{C}$ une catégorie avec un produit tensoriel symétrique fermée. Un foncteur $(-)^* : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ est un *foncteur dualisant* s'il existe un isomorphisme $\mathsf{d}_{A,B}$: $A \multimap B \to B^* \multimap A^*$, naturel en A et B, tel que le diagramme

$$\begin{tikzcd}
(A \multimap B) \otimes (B \multimap C) \arrow[r, "\mathsf{c}_{A,B,C}"] \arrow[d, "\mathsf{d}_{A,B} \otimes \mathsf{d}_{B,C}"'] \arrow[dr, phantom, "(D)"] & A \multimap C \arrow[d, "\mathsf{d}_{A,C}"] \\
(B^* \multimap A^*) \otimes (C^* \multimap B^*) \arrow[r, "\mathsf{c}_{C^*,B^*,A^*} \circ \sigma_{B^* \multimap A^*, C^* \multimap B^*}"'] & C^* \multimap A^*
\end{tikzcd}$$

soit commutatif pour tout objets A, B et C dans $\mathscr{C}$.

Definition 8.5. Une catégorie $*$-*autonome* est une catégorie monoïdale fermée munie d'un foncteur dualisant.

Definition 8.6. Soit $\mathscr{C}$ une catégorie avec un produit tensoriel symétrique fermée. Un objet $\perp$ est dit *dualisant* si pour tout objet A dans $\mathscr{C}$, le morphisme canonique $A \to (A \multimap \perp) \multimap \perp$ est un isomorphisme.

Definition 8.7. Une *catégorie* $*$-*autonome* est une catégorie monoïdale symétrique fermée munie d'un objet dualisant.

Proposition 8.8. *Les définitions 8.5 et 8.7 sont équivalentes.*

8.3 Catégories $*$-autonomes sans unité

Soient $\mathscr{C}$ une catégorie et $\mathcal{Y}$ le foncteur de Yoneda contravariant $\mathcal{Y} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ défini par $\mathcal{Y}(A) = \mathsf{Hom}_{\mathscr{C}}(A, -)$ pour A dans $\mathscr{C}$. On notera également h^A l'objet $\mathcal{Y}(A)$. Le lemme de Yoneda affirme que le foncteur $\mathcal{Y}$ est pleinement fidèle et justifie donc la définition suivante.

Definition 8.9. Soit $\mathscr{C}$ une catégorie. On appellera objet virtuel tout objet de $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$.

Definition 8.10. Soit $\mathscr{C}$ une catégorie avec un produit tensoriel fermée. On définit le *produit tensoriel virtuel* $(-) \otimes_v (-)$ par

$$\begin{array}{rccc} (-) \otimes_v (-) : & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \times \mathscr{C}^{\mathrm{op}} & \to & \mathsf{Fun}(\mathscr{C}, \mathsf{Set}) \\ & (\mathbb{A}, X) & \mapsto & \mathbb{A} \circ H^X \end{array}$$

pour tout objet virtuel $\mathbb{A}$ de $\mathscr{C}$ et tout objet X de $\mathscr{C}$.

Definition 8.11. Une *catégorie monoïdale symétrique sans unité* est une catégorie avec un produit tensoriel fermée $\mathscr{C}$ munie d'un objet virtuel $\mathbb{I}$ et d'un isomorphisme

$$\tilde{\lambda} : \mathbb{I} \otimes_v A \to h^A$$

naturel en $A \in \mathscr{C}$ tel que

$$\begin{tikzcd} \mathbb{I} \otimes_v (B \otimes C) \arrow[r, "\tilde{\alpha}_{\mathbb{I},B,C}"] \arrow[d, "\tilde{\lambda}_{B\otimes C}"'] \arrow[dr, phantom, "(Tv)"] & (\mathbb{I} \otimes_v B) \otimes_v C \arrow[d, "\tilde{\lambda}_{B\otimes_v C}"] \\ h^{B\otimes C} \arrow[r, "\Theta_{B,C}"'] & h^B \otimes_v C \end{tikzcd}$$

soit commutatif pour tout objets B, C de $\mathscr{C}$, où

$$\tilde{\alpha}_{\mathbb{I},B,C}(Z) = \mathbb{I}(\psi_{B,C}(Z))$$

pour tout B, C, Z dans $\mathscr{C}$ (voir l'équation 9 pour la définition de ψ).

Si de plus $\mathscr{C}$ est munie d'un foncteur dualisant, on dira que $\mathscr{C}$ est une *catégorie $*$-autonome sans unité.*

Proposition 8.12.

1. *Soit $\mathscr{C}$ une catégorie $*$-autonome sans unité d'unité virtuelle $\mathbb{I}$. Si $\mathbb{I}$ est représentable par un objet I de $\mathscr{C}$, alors $\mathscr{C}$ est une catégorie $*$-autonome d'unité I.*
2. *Soit $\mathscr{C}$ une catégorie $*$-autonome d'unité I. Alors $\mathscr{C}$ est une catégorie $*$-autonome sans unité d'unité virtuelle h^I.*

8.4 La catégorie des réseaux de preuve de MLL^-

L'ensemble des formules de MLL^- est défini par récurrence de la manière suivante :

$$\mathscr{F} := \mathscr{A} \mid \mathscr{A}^* \mid \mathscr{F} \parr \mathscr{F} \mid \mathscr{F} \otimes \mathscr{F}$$

où $\mathscr{A} = \{a, b, c, \ldots\}$ est un ensemble dénombrable de variables propositionnelles, et $\mathscr{A}^* = \{a^*, b^*, c^*, \ldots\}$ est l'ensemble de leurs duaux. Nous appellerons *atomes* les éléments de $\mathscr{A} \cup \mathscr{A}^*$.

Le dual d'une formule est défini par récurrence :

$$a^{**} = a \qquad (A \otimes B)^* = B^* \parr A^* \qquad (A \parr B)^* = B^* \otimes A^*.$$

L'implication linéaire est définie par $A \multimap B := A^* \parr B$.

Les règles d'inférence de MLL^- sont les suivantes :

$$\begin{gathered} id\,\frac{}{\vdash A^*, A} \qquad exch\,\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta} \qquad \parr\,\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \parr B, \Delta} \\ \otimes\,\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \qquad cut\,\frac{\vdash \Gamma, A \quad \vdash A^*, \Delta}{\vdash \Gamma, \Delta} \end{gathered} \tag{10}$$

Les preuves en calcul des séquents contiennent beaucoup de détails inutiles. La théorie des réseaux de preuve de Girard introduite dans [Gir87] permet de représenter les preuves d'une manière qui identifie deux preuves ne différant que par des règles non essentielles.

Dans ce papier, nous allons utiliser la notion de réseaux de preuve de MLL^- où les preuves de MLL^- sont représentées par leurs graphes de cohérence ainsi que des liens axiomes (voir [Str06] et [Hug05]).

Ces réseaux de preuve forment une catégorie dont les objets sont les formules construites à partir des atomes et dont les flèches sont les réseaux de preuve sans coupures. Pour chaque formule A, l'identité de A est le réseau de preuve $\vdash A^*, A$ avec un lien axiome ente A^* et A. Si on a deux réseaux de preuve f de $\vdash A^*, B$ et g de $\vdash B^*, C$, la composition $g \circ f$ s'obtient en appliquant l'élimination des coupures à $A^*, B \,\varobar\, B^*, C$, où $\varobar$ est le connecteur de coupure. L'associativité de la composition résulte de la forte normalisation de l'élimination des coupures.

Nous noterons $\mathsf{PN}^-(\mathscr{A})$ cette catégorie.

Proposition 8.13. *Pour tout ensemble de variables propositionnelles $\mathscr{A}$, la catégorie $\mathsf{PN}^-(\mathscr{A})$ est une catégorie $*$-autonome sans unité.*

8.5 Catégories $*$-autonomes libres

Definition 8.14. Soit $\mathscr{C}$ et $\mathscr{C}'$ deux catégories $*$-autonomes. Un foncteur $F : \mathscr{C} \to \mathscr{C}'$ est dit *$*$-autonome strict* si :

1. on a $F(I) = I'$, où I et I' sont les unités de $\mathscr{C}$ et $\mathscr{C}'$;

2. on a $F(A \otimes B) = F(A) \otimes F(B)$ pour tout objets A et B dans $\mathscr{C}$;
3. on a $F(A \multimap B) = F(A) \multimap F(B)$ pour tout objets A et B dans $\mathscr{C}$;
4. on a $F(\perp) = \perp'$, où $\perp$ et $\perp'$ sont les objets dualisants de $\mathscr{C}$ et $\mathscr{C}'$.

Definition 8.15. Soit $\mathscr{C}$ et $\mathscr{C}'$ deux catégories $*$-autonomes sans unité. Un foncteur $F : \mathscr{C} \to \mathscr{C}'$ est dit *$*$-autonome strict sans unité* si :

1. les foncteurs $\mathbb{I}' \circ F$ et $\mathbb{I}$ sont isomorphes, où $\mathbb{I}$ et $\mathbb{I}'$ sont les unités virtuelles de $\mathscr{C}$ et $\mathscr{C}'$;
2. on a $F(A \otimes B) = F(A) \otimes F(B)$ pour tout objets A et B dans $\mathscr{C}$;
3. on a $F(A \multimap B) = F(A) \multimap F(B)$ pour tout objets A et B dans $\mathscr{C}$.

Notons Obj le foncteur qui a une catégorie $*$-autonome sans unité associe son ensemble sous-jacent.

Definition 8.16. Soit S un ensemble. La catégorie $*$-autonome sans unité libre sur S est une catégorie $*$-autonome sans unité $\mathscr{C}$ munie d'une application $S \to \mathsf{Obj}(\mathscr{C})$ telle que pour toute catégorie $*$-autonome sans unité $\mathscr{D}$ et toute application $S \to \mathsf{Obj}(\mathscr{D})$, il existe un unique foncteur $*$-autonome strict sans unité tel que le diagramme

$$\begin{tikzcd} & \mathsf{Obj}(\mathscr{C}) \arrow[dd, "\mathsf{Obj}(f)"] \\ S \arrow[ur] \arrow[dr] & \\ & \mathsf{Obj}(\mathscr{D}) \end{tikzcd}$$

soit commutatif.

On voudrait que le catégorie $\mathsf{PN}^-(\mathscr{A})$ soit le catégorie $*$-autonome sans unité libre sur l'ensemble $\mathscr{A}$. Avec la notion de foncteurs $*$-autonomes stricts sans unité définie ci-dessus, ce résultat est faux. En effet, il n'existe pas de foncteur $*$-autonome strict sans unité de la catégorie $\mathsf{PN}^-(\mathscr{A})$ vers la catégorie terminale (un unique objet et une unique flèche) qui est pourtant $*$-autonome sans unité. Ce problème est dû à la condition de préservation des unités virtuelles dans la définition des foncteurs $*$-autonomes stricts sans unité. Celle-ci semble ne pas être la bonne et il faudrait la modifier.

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